# STEP 2013 Paper 2

Unofficial Worked Answers

17th February 2021

## Contents




A huge thanks to Adithya, Artyom and Gilad, who helped me with corrections and alternative solutions.

# 1 Curves and Intersections

(i) Two curves 'touching' at a given point means that they have the same height ($y$ position) and same first derivative. Hence,

$$mx = \ln x; \quad m = \frac{1}{x}$$

We solve the two equations simultaneously, substituting the second into the first, giving

$$\begin{aligned} \frac{1}{x} \cdot x &= \ln x \\ 1 &= \ln x \\ e &= x \\ \implies m &= \frac{1}{e} \end{aligned}$$

Note that $x$ cannot be zero, since $\ln x$ is undefined at this point. Hence dividing by $x$ is allowed. For the second part, 'intersecting' means simply that they have the same height ($y$ position) at a given point. So we have

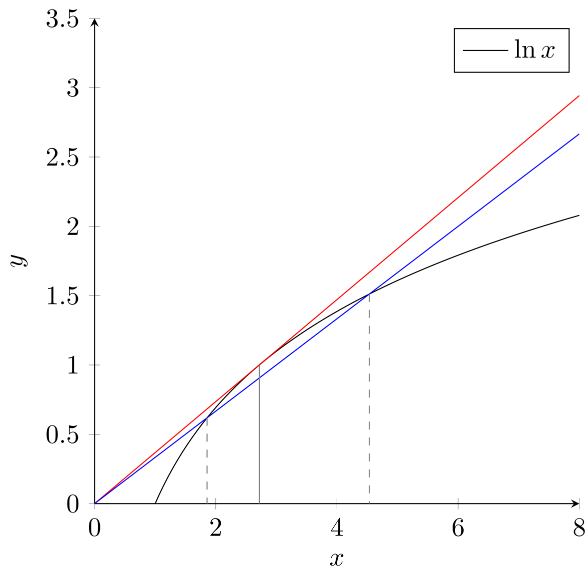
$$ma = \ln a; \quad mb = \ln b$$

We are trying to show that $a^b = b^a$, so we need to somehow remove the $m$ variable from the equations. A simple way to remove a variable from two equations is to make that variable the subject of both equations.

$$m = \frac{\ln a}{a}; \quad m = \frac{\ln b}{b}$$

Now, we can set the equations equal to each other.

$$\begin{aligned} \frac{\ln a}{a} &= \frac{\ln b}{b} \\ b \ln a &= a \ln b \\ a^b &= b^a \end{aligned}$$




The red line is the tangent that we calculated in the first part. The blue line intersects the curve at $x = a$ and $x = b$. We can see that if $a < b$, we must have a shallower gradient than $e^{-1}$. We can then see that the intersection points must lie either side of $e$.

(ii) Similarly to part (i), we have

$$mp + c = \ln p; \quad mq + c = \ln q$$

Hence,

$$\frac{\ln p - c}{p} = m = \frac{\ln q - c}{q}$$

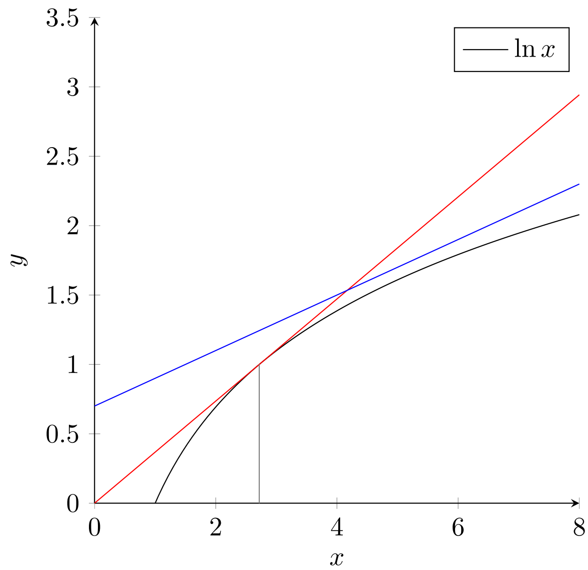
We can expand to get

$$\frac{\ln p}{p} - \frac{c}{p} = \frac{\ln q}{q} - \frac{c}{q}$$
$$\frac{\ln p}{p} + \frac{c}{q} = \frac{\ln q}{q} + \frac{c}{p}$$

But $p < q$ hence $\frac{1}{p} > \frac{1}{q}$. So

$$\frac{\ln p}{p} > \frac{\ln q}{q}$$
$$q \ln p > p \ln q$$
$$p^q > q^p$$

(iii) We can just use part (ii) of the question. Both points are on the curve $y = \ln x$.

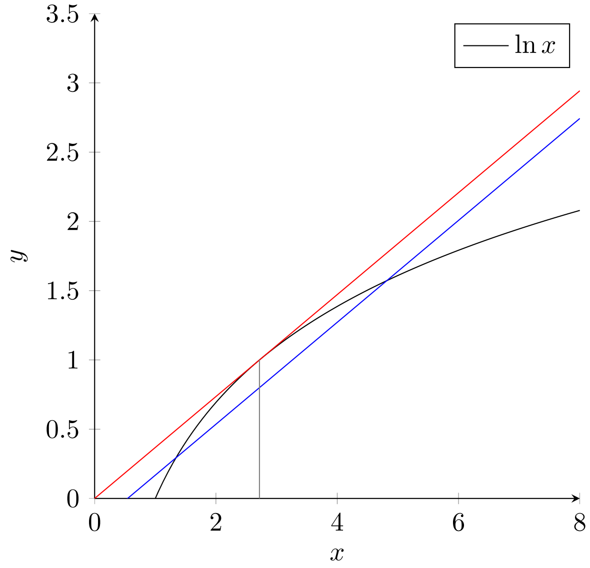



In order to touch both points which are to the right of $e$, the $y$-intercept must be positive. Now, let $p = e$ and $q = \pi$, since $e < \pi$. Then the $y$-intercept is positive, and part (ii) of the question applies. We can deduce that $e^\pi > \pi^e$.

(iv) You should recognise the $\frac{\ln q - \ln p}{q-p}$ term as the gradient of the line joining two points.

$$y = \ln x \implies \frac{\ln q - \ln p}{q - p} = \frac{\Delta y}{\Delta x}$$

So the gradient of the line joining $(p, \ln p)$ and $(q, \ln q)$ is given by this fraction $\frac{\Delta y}{\Delta x}$, which we have been told is $e^{-1}$. Recall that in part (i) of this question, we proved that the line tangent to $\ln x$ passing through the origin had gradient $e^{-1}$. Hence, in order for a line of that gradient to intersect two points, it must have been moved down.




Therefore, the equation of this blue line is $e^{-1}x + c$ where $c$ is negative. We can use the same logic as in part (ii), but this time solving for negative $c$.

$$\begin{aligned}\frac{\ln p}{p} - \frac{c}{p} &= \frac{\ln q}{q} - \frac{c}{q} \\ \frac{\ln p}{p} + \frac{c}{q} &= \frac{\ln q}{q} + \frac{c}{p}\end{aligned}$$

We know that $p > q$ hence $\frac{1}{p} < \frac{1}{q}$. So

$$\begin{aligned}\frac{\ln p}{p} &< \frac{\ln q}{q} \\ q \ln p &< p \ln q \\ p^q &< q^p\end{aligned}$$

as required.

# 2 Integration

(i) We can compare the two integrals to notice a few important similarities:

- they both range over the same limits;
- the powers on each term are the same, but swapped;
- $1 - x$ and $x$ both range over the interval $[0, 1]$ in opposite directions.

From this, we might guess that in the first integral we need to substitute $x$ for $1 - u$, which would mean $1 - x$ is replaced with $u$. Using this substitution, $\mathrm{d}x = -\,\mathrm{d}u$, and we can find

$$\begin{aligned}\int_0^1 x^{n-1}(1-x)^n \,\mathrm{d}x &= -\int_1^0 (1-u)^{n-1} u^n \,\mathrm{d}u \\ &= \int_0^1 (1-u)^{n-1} u^n \,\mathrm{d}u \\ &= \int_0^1 (1-x)^{n-1} x^n \,\mathrm{d}x \qquad (*)\end{aligned}$$

Remember that the variable that we're integrating over doesn't appear in the final answer. So we are free to rename that variable to whatever we like. For the next part, since we're trying to work with twice the left hand side of $(*)$, it seems like a logical step to instead add the left hand side of $(*)$ to the right hand side of $(*)$. This allows us to use what we've already deduced in the first part. This is a hallmark of STEP questions: whenever you get the opportunity to use something you've already worked out, take it!

$$2\int_0^1 x^{n-1}(1-x)^n \,\mathrm{d}x = \int_0^1 x^{n-1}(1-x)^n \,\mathrm{d}x + \int_0^1 (1-x)^{n-1}x^n \,\mathrm{d}x$$

Since the limits of integration are the same, we can combine the two integrals and factorise out the common terms.

$$\begin{aligned}
\int_0^1 \left(x^{n-1}(1-x)^n + (1-x)^{n-1}x^n\right) \mathrm{d}x &= \int_0^1 x^{n-1}(1-x)^{n-1}\left((1-x)+x\right) \mathrm{d}x \\
&= \int_0^1 x^{n-1}(1-x)^{n-1}\,(1)\,\mathrm{d}x \\
&= I_{n-1}
\end{aligned}$$

Comparing our target to the equation at the top for $I_n$, we can see that one of the powers has increased and the other has decreased. This reminds us of the technique of integration by parts, where one term is differentiated and the other is integrated. Trying this technique, we get

$$\begin{aligned}
I_n &= \int_0^1 x^n(1-x)^n \,\mathrm{d}x \\
&= \left[\frac{-1}{n+1}x^n(1-x)^{n+1}\right]_0^1 - \int_0^1 nx^{n-1}\cdot\frac{-1}{n+1}(1-x)^{n+1}\,\mathrm{d}x \\
&= 0 + \int_0^1 nx^{n-1}\cdot\frac{1}{n+1}(1-x)^{n+1}\,\mathrm{d}x \\
&= \frac{n}{n+1}\int_0^1 x^{n-1}(1-x)^{n+1}\,\mathrm{d}x
\end{aligned}$$

Now, to establish the final result, we notice that our result for $I_{n-1}$ contains an integral over $x^{n-1}(1-x)^n$, and our result for $I_n$ contains an integral over $x^{n-1}(1-x)^{n+1}$. In order to combine these, we need to therefore take away that extra factor from the $I_n$ integral somehow. We can expand out this bracket to create a sum of two integrals.

$$\begin{aligned}
I_n &= \frac{n}{n+1}\int_0^1 x^{n-1}(1-x)^{n+1}\,\mathrm{d}x \\
&= \frac{n}{n+1}\int_0^1 (1-x)x^{n-1}(1-x)^n\,\mathrm{d}x \\
&= \frac{n}{n+1}\int_0^1 x^{n-1}(1-x)^n\,\mathrm{d}x - \frac{n}{n+1}\int_0^1 x^n(1-x)^n\,\mathrm{d}x \\
&= \frac{n}{n+1}\cdot\frac{1}{2}I_{n-1} - \frac{n}{n+1}\cdot I_n \\
\implies I_n + \frac{n}{n+1}I_n &= \frac{n}{2(n+1)}I_{n-1} \\
\frac{2n+1}{n+1}I_n &= \frac{n}{2(n+1)}I_{n-1} \\
I_n &= \frac{n(n+1)}{2(n+1)(2n+1)}I_{n-1} \\
I_n &= \frac{n}{2(2n+1)}I_{n-1}
\end{aligned}$$

(ii) We have now established a relationship that allows us to compute $I_n$ given $I_{n-1}$. We are also told that $n$ is only a positive integer. These two ideas remind us of proof by induction. For proof by induction, we must always consider a base case. Here, a simple base case could be $n = 1$. We can compute the integral in this case directly from the definition:

$$
\begin{aligned}
I_1 &= \int_0^1 x^1(1-x)^1\,\mathrm{d}x \\
&= \int_0^1 (x - x^2)\,\mathrm{d}x \\
&= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 \\
&= \frac{1}{2} - \frac{1}{3} \\
&= \frac{1}{6} \\
&= \frac{(1!)^2}{(2\cdot 1+1)!}
\end{aligned}
$$

as required. Now, we can make an inductive step. Given that $I_n = \frac{(n!)^2}{(2n+1)!}$, we want to show that $I_{n+1} = \frac{((n+1)!)^2}{(2(n+1)+1)!}$. We will use the result we proved in part (i) to help us move from the $n$ case to the $n+1$ case.

$$
\begin{aligned}
I_{n+1} &= \frac{n+1}{2(2(n+1)+1)} I_n \\
&= \frac{n+1}{2(2(n+1)+1)} \frac{(n!)^2}{(2n+1)!} \\
&= \frac{(n+1)\cdot(n!)^2}{2(2n+3)\cdot(2n+1)!} \\
&= \frac{(n+1)!\cdot n!}{2(2n+3)!/(2n-2)} \\
&= \frac{(n+1)!\cdot n!}{(2n+3)!/(n-1)} \\
&= \frac{(n+1)!\cdot n!\cdot(n+1)}{(2n+3)!} \\
&= \frac{(n+1)!\cdot(n+1)!}{(2n+3)!}
\end{aligned}
$$

as required. Remember that we can extend or reduce factorials by a term, by multiplying or dividing by that term, for instance $n! = (n-1)!\cdot n = (n+1)!/n$.

(iii) For this last part, we need to evaluate $I_{\frac{1}{2}}$ directly. We can't use part (ii) because $\frac{1}{2}$ isn't a positive integer.

$$
I_{\frac{1}{2}} = \int_0^1 \sqrt{x(1-x)}\,\mathrm{d}x
$$

With the substitution $x = \sin^2\theta$, we have $\mathrm{d}x = 2\sin\theta\cos\theta\,\mathrm{d}\theta$. The end points of the integral become $\theta = 0$ and $\theta = \frac{\pi}{2}$. Substituting into the integral, applying the double-angle formula for sine, we have

$$
\begin{aligned}
I_{\frac{1}{2}} &= \int_0^{\frac{\pi}{2}} \sqrt{\sin^2\theta(1-\sin^2\theta)}\cdot 2\sin\theta\cos\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \sqrt{\sin^2\theta\cos^2\theta}\cdot\sin 2\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\cdot\sin 2\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\cdot\sin 2\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\,\mathrm{d}\theta
\end{aligned}
$$

We can now apply the double-angle formula for cosine, to get

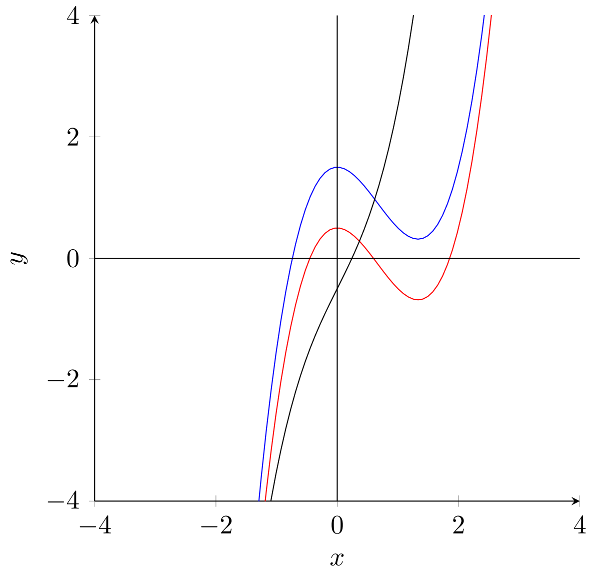
$$
\begin{aligned}
I_{\frac{1}{2}} &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\,\mathrm{d}\theta \\
&= \frac{1}{4}\left[(\theta - \frac{1}{4}\sin 4\theta)\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{4}\cdot\frac{\pi}{2} \\
&= \frac{\pi}{8}
\end{aligned}
$$

We can now use the result in part (i), since none of this reasoning required $n$ to be an integer.

$$
\begin{aligned}
I_{\frac{3}{2}} &= \frac{3/2}{2(2(3/2)+1)}\cdot\frac{\pi}{8} \\
&= \frac{3}{4(3+1)}\cdot\frac{\pi}{8} \\
&= \frac{3}{16}\cdot\frac{\pi}{8} \\
&= \frac{3\pi}{128}
\end{aligned}
$$

## 3 Cubic Equation

(i) The $y$-intercept is positive, and the cubic is positive.




The red curve illustrates the three-root case. By moving the curve upwards, we get a one-root case, depicted in blue. The white curve shows the case where the graph has no turning points, which always gives a single root.

(ii) Since the cubic is positive, as $x \to -\infty$, $f(x) \to -\infty$. Since all three roots lie to the right hand side of the origin, we must have $c < 0$ as the graph has not yet crossed the $x$-axis. Now, as we have seen in the sketch, the graph must have two turning points in order to have three distinct roots. The turning points can be found where the derivative of the function is zero.

$$3x^2 + 6ax + 3b = 0 \implies x^2 + 2ax + b = 0 \qquad (\dagger)$$

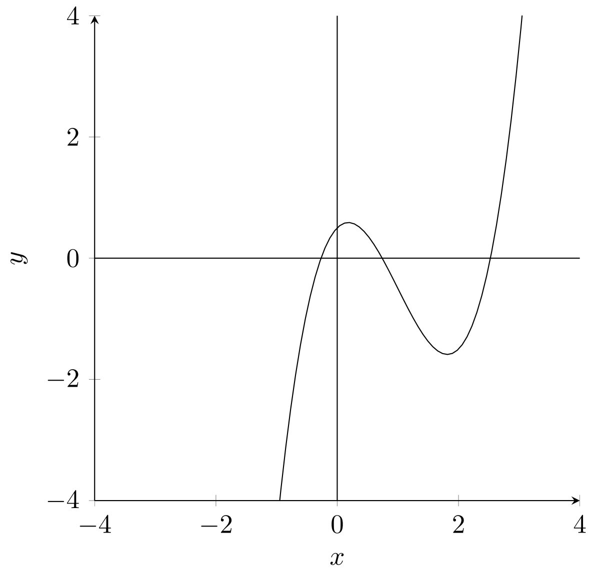
We need (†) to have two solutions, so that the cubic has two turning points. A quadratic has two solutions if and only if its discriminant is greater than zero. Hence,

$$(2a)^2 - 4 \cdot 1 \cdot b > 0 \implies a^2 > b$$

Both turning points must be at $x > 0$. So the solutions to (†) must be at $x > 0$. If $b \leq 0$, then one solution would be negative and one would be positive, so $b$ must be greater than zero. If $a \geq 0$, then $x^2 + 2ax + b$ is always positive if $x$ is positive, but since both our roots are for positive $x$, we must have $a < 0$.

(iii) This is a positive cubic. Since $c > 0$, the $y$-intercept is positive. Since $ab < 0$, we know that either $a$ is positive and $b$ is negative, or $a$ is negative and $b$ is positive. Once again, let us consider the turning points of this cubic. Once again, the turning points occur as solutions to (†), but this time the signs of $a$, $b$, $c$ may be different.

If $b < 0$, then (†) has both a positive and a negative solution for $x$ (this can be seen by simply graphing the function, since it is a positive quadratic). If $a < 0$ (and therefore $b > 0$), the solutions to (†), if any, must be positive (again this can be seen by drawing a graph, or differentiating (†) and showing that the turning point on this graph is positive). So in either case, the cubic has at least one turning point for some positive $x$. So the graph looks something like this:




Two roots are positive, and one root is negative.

(iv) For this last part, note that the conditions (∗) are independent of the value of $c$. Since $c$ represents the vertical translation of the graph, we can simply pick a graph that satisfies (∗), then translate it very high so that the turning points are well above the $x$-axis. This would give a graph with only one root that satisfies (∗). A simple equation that satisfies (∗) is

$$a = -2; \quad b = 1; \quad c = -1$$

The turning points of this function have a $y$ position somewhere between $-100$ and $100$ (not a very good bound, but a bound nonetheless), so the graph

$$x^3 - 6x^2 + 3x - 101$$

certainly has only one real root, and satisfies all conditions in (∗).

# 4 Locus

The circle with unit radius has equation

$$x^2 + y^2 = 1$$

and the relevant line has equation

$$y = b(x - a)$$

We can substitute the second equation into the first, giving

$$x^2 + b^2(x-a)^2 = 1$$

This is a quadratic for $x$, where the two solutions give the $x$-coordinates of $P$ and $Q$. Note, however, that the $x$ coordinate of the turning point of any quadratic is the mean of the two solutions. So we don't actually need to solve the quadratic, we only need its midpoint. So we can take the derivative of this quadratic, and solve that instead.

$$2x + 2b^2(x-a) = 0 \implies \left(1+b^2\right)x = ab^2 \implies x = \frac{ab^2}{1+b^2}$$

$M$ is on the line, hence the $y$ coordinate is given by

$$y = b\left(\frac{ab^2}{1+b^2} - a\right) = b\left(\frac{ab^2 - a - ab^2}{1+b^2}\right) = \frac{-ab}{1+b^2}$$

(i) We want to eliminate the independent variable $a$ from the equation, so that we can get an expression that is true regardless of the value of $a$. So let $a$ be the subject of our equations for $x$ and $y$.

$$a = \frac{x(1+b^2)}{b^2}; \quad a = \frac{-y(1+b^2)}{b}$$

Combining the equations, we get

$$\begin{aligned}\frac{x(1+b^2)}{b^2} &= \frac{-y(1+b^2)}{b}\\ \frac{x}{b^2} &= \frac{-y}{b}\\ x &= -by\end{aligned}$$

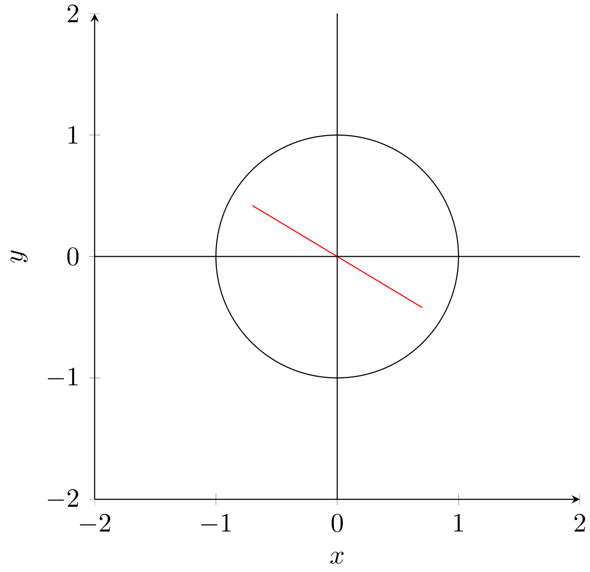
This is an equation for a line that passes through the origin. Now, we need to work out the length of the line. We can do this using Pythagoras' Theorem. The height of the triangle is the difference in $y$ from the start point to the end point, and the width of the triangle is the difference in $x$. Since $a$ ranges from $-1$ to 1, we have

$$\Delta x = \frac{2b^2}{1+b^2}; \quad \Delta y = \frac{2b}{1+b^2}$$

Hence, the length of the line $\ell$ is given by

$$\begin{aligned}\ell^2 &= \Delta x^2 + \Delta y^2\\ &= \left(\frac{2b^2}{1+b^2}\right)^2 + \left(\frac{2b}{1+b^2}\right)^2\\ &= \frac{4b^4+4b^2}{(1+b^2)^2}\\ &= \frac{4b^2(1+b^2)}{(1+b^2)^2}\\ &= \frac{4b^2}{1+b^2}\\ \implies \ell &= \frac{2b}{\sqrt{1+b^2}}\end{aligned}$$

Notice that $\ell < 2$ for $b > 0$, since the denominator is always larger than $b$. Therefore, the line $M$ sits inside the circle (of diameter 2) and does not intersect it. The locus of $M$ is shown in red in the following diagram.




Alternatively, we could use the substitution $b = \tan t$ for some $-\frac{\pi}{2} < t < \frac{\pi}{2}$. This is motivated by the fact that $1 + \tan^2 t = \sec^2 t$, and since $\sec^2$ will be on the denominator, we end up with a simple $\cos^2$ factor on the numerator. We then have

$$\begin{aligned}\Delta x &= \frac{2\tan^2 t}{\sec^2 t} = 2\tan^2 t\cos^2 t = 2\sin^2 t\\ \Delta y &= \frac{2\tan t}{\sec^2 t} = 2\tan t\cos^2 t = 2\sin t\cos t\end{aligned}$$

Therefore,

$$\begin{aligned}\ell^2 &= \Delta x^2 + \Delta y^2\\ &= 4\sin^4 t + 4\sin^2 t\cos^2 t\\ &= 4\sin^2 t\left(\sin^2 t + \cos^2 t\right)\\ &= 4\sin^2 t\\ \therefore \ell &= 2\sin t\end{aligned}$$

Since $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $-2 < 2\sin t < 2$ so it sits inside the circle of diameter 2.

(ii) We need to solve these parts individually.

(a) Since $b$ is the independent variable, let us make $b$ the subject. In fact, it appears easiest to make $b^2$ the subject in the equation for $x$.

$$x + b^2 x = ab^2 \implies (a - x)b^2 = x \implies b^2 = \frac{x}{a - x}$$

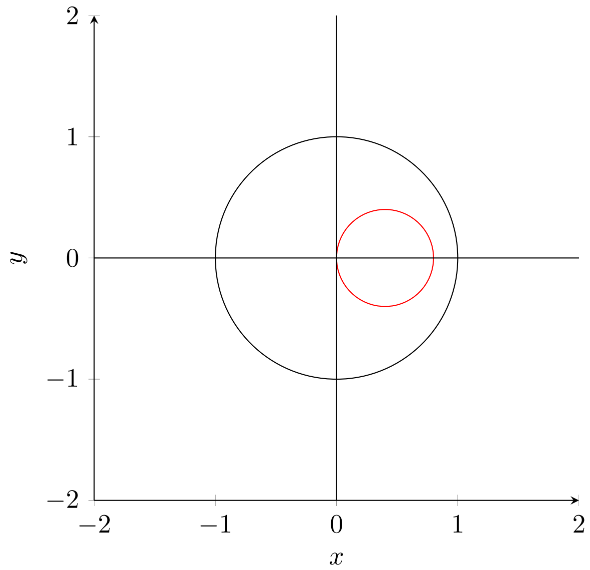
We can substitute into the expression for $y$, noting that we need to consider both positive and negative square roots of $b^2$, to give

$$\begin{aligned}y &= \frac{\pm a\sqrt{\frac{x}{a-x}}}{1 + \frac{x}{a-x}}\\ &= \pm\frac{a\sqrt{\frac{x}{a-x}}(a - x)}{a - x + x}\\ &= \pm\frac{a\sqrt{\frac{x}{a-x}}(a - x)}{a}\\ &= \pm\sqrt{\frac{x}{a - x}}(a - x)\\ &= \pm\sqrt{x(a - x)}\end{aligned}$$

It is easiest to identify this equation by squaring both sides.

$$\begin{aligned} y^2 &= x(a-x) \\ x^2 - ax + y^2 &= 0 \\ \left(x - \frac{1}{2}a\right)^2 + y^2 &= \frac{a^2}{4} \end{aligned}$$

This forms a circle, centred at $\left(\frac{a}{2}, 0\right)$ with radius $\frac{a}{2}$. The following diagram shows the case $a = 0.8$.




(b) We can substitute $ab = 1$ into our equations for $x$ and $y$ to give

$$x = \frac{b}{1+b^2}; \quad y = \frac{-1}{1+b^2}$$

Then $b$ is the independent parameter, which varies with $0 < b \le 1$. We can solve this like before, by making $b$ the subject. Here, it appears that manipulating the equation for $y$ will be easier since $b$ appears only once.

$$y + b^2 y = -1 \implies b^2 = \frac{-1-y}{y}$$

We can substitute this intok the equation for $x$ to give

$$\begin{aligned} x &= \frac{\sqrt{\frac{-1-y}{y}}}{1 + \frac{-1-y}{y}} \\ &= \frac{\sqrt{y(-1-y)}}{y-1-y} \\ &= -\sqrt{-y(1+y)} \end{aligned}$$

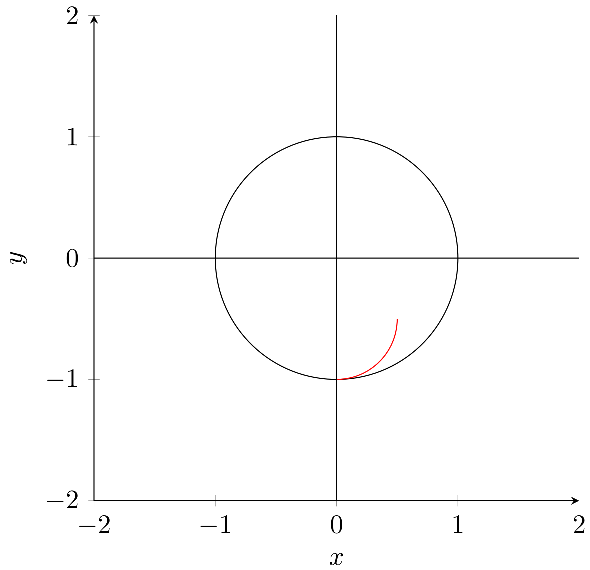
Squaring both sides and completing the square we can see

$$\begin{aligned} x^2 &= -y(1+y) \\ x^2 + y^2 + y &= 0 \\ x^2 + \left(y + \frac{1}{2}\right)^2 &= \frac{1}{4} \end{aligned}$$

This is the circle centred at $\left(0, -\frac{1}{2}\right)$ with radius $\frac{1}{2}$. Note, however, that $b$ is restricted to the range $0 < b \le 1$, so perhaps not the entire circle is covered by this locus. We can compute that the end points have coordinates

$$b = 0 \implies (x,y) = (0,-1); \quad b = 1 \implies (x,y) = \left(\frac{1}{2}, \frac{-1}{2}\right)$$

We can see that the locus actually traces out the bottom-right quarter-circle. We know that it traces out this quarter circle (and not the other three quarters) because $x$ is always positive for $0 < b \leq 1$, which does not hold for the left half of the circle.




## 5 Functions and Derivatives

(i) Since the equation $f(x) = f(1-x)$ is true for all $x$, we can differentiate both sides.

$$\begin{aligned} f(x) &= f(1-x) \\ f'(x) &= f'(1-x) \cdot -1 \\ f'(x) + f'(1-x) &= 0 \end{aligned}$$

We can substitute $x = \frac{1}{2}$ to get

$$f'\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) = 0 \implies f'\left(\frac{1}{2}\right) = 0$$

Notice that the number $\frac{1}{2}$ was special here because $\frac{1}{2} = 1 - \frac{1}{2}$. We can guess that the next part of the question also relies on differentiating the identity, since the results have $f'$ in them.

$$\begin{aligned} f(x) &= f\left(\frac{1}{x}\right) \\ f'(x) &= f'\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} \\ f'(x) + \frac{1}{x^2} f'\left(\frac{1}{x}\right) &= 0 \end{aligned}$$

Substituting $x = -1$, we have

$$f'(-1) + \frac{1}{(-1)^2} f'\left(\frac{1}{-1}\right) = 0 \implies f'(-1) + f'(-1) = 0 \implies f'(-1) = 0$$

Note that $-1$ specifically worked here because $-1 = \frac{1}{-1}$. 2 isn't a special number here, but we know how to relate it to a previous result: we know already that $f'\left(\frac{1}{2}\right) = 0$. So we can use the relationship we deduced for reciprocals, and substitute in 2 (or $\frac{1}{2}$).

$$f'(2) + \frac{1}{2^2} f'\left(\frac{1}{2}\right) = 0 \implies f'(2) = -\frac{1}{4} f'\left(\frac{1}{2}\right) = 0$$

(ii) We can show both of these things directly, by substituting in $\frac{1}{x}$ and $1 - x$ into $f$.

$$f\left(\frac{1}{x}\right) = \frac{(x^{-2} - x^{-1} + 1)^3}{(x^{-2} - x^{-1})^2}$$

We would like to remove all of the negative powers, so we multiply the top and bottom of the function by the biggest negative power (after expanding), which is $x^6$. We don't need to worry about dividing by zero, since $f$ is defined only for $x \neq 0$.

$$\begin{aligned} f\left(\frac{1}{x}\right) &= \frac{(1-x+x^2)^3}{(x-x^2)^2} \\ &= \frac{(x^2-x+1)^3}{(-(x^2-x))^2} \\ &= \frac{(x^2-x+1)^3}{(x^2-x)^2} \\ &= f(x) \end{aligned}$$

Notice that squaring a negative removes the negative sign, which is why we could rearrange the denominator. For $1-x$, we have

$$\begin{aligned} f(1-x) &= \frac{((1-x)^2-(1-x)+1)^3}{((1-x)^2-(1-x))^2} \\ &= \frac{(x^2-2x+1-1+x+1)^3}{(x^2-2x+1-1+x)^2} \\ &= \frac{(x^2-x+1)^3}{(x^2-x)^2} \\ &= f(x) \end{aligned}$$

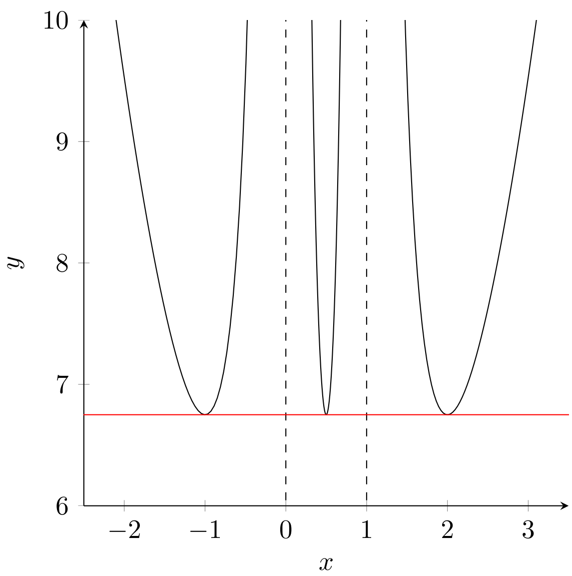
Make sure you don't forget (or add in) negative signs in lengthy algebraic manipulation. Now, we have proved that $f$ satisfies both conditions we met in part (i), so in particular it has turning points at $x = -1, \frac{1}{2}, 2$. We are told that these are the only turning points, so all that we really need in order to sketch the function is to see how it behaves as $x$ tends towards critical points of the function and infinity.

Note that the function's denominator vanishes to 0 from the positive direction as $x \to 0$ and $x \to 1$. In both of these cases, we can see that the numerator is positive either side of the critical points. Therefore, we are dividing a positive value by a very small positive value, giving a very large positive value as a result. So near $x = 0$ and $x = 1$, the function tends to $+\infty$.

As $x$ tends to either positive or negative infinity, the $x^6$ term on the numerator (when expanded) will dominate the expression, showing that $f(x)$ will tend to positive infinity in both cases.

We can evaluate the nature of each turning point heuristically. For example, the turning point at $x = -1$ must be a minimum point, since to the left ($x \to -\infty$) the function tends to $+\infty$, and to the right ($x \to 0$) the function also tends to $+\infty$. So the function is increasing on both sides of the point, so it is a minimum. It turns out that the same holds for each point, and hence they are all minimum points.

Now, we can evaluate the function at each turning point. Each turning point has $f(x) = \frac{27}{4}$ (the calculations are omitted here, since they are not complicated). Now, with this knowledge, we can sketch the graph. The dashed lines are the asymptotes, and the red line is the line $y = \frac{27}{4}$, which will be useful for the next part of the question.



(iii) By considering the graph we have just drawn, it is clear that the only solutions to $f(x) = \frac{27}{4}$ are the turning points: $x = -1, \frac{1}{2}, 2$. We can also see then that the ranges of $x$ for which $f(x) > \frac{27}{4}$ are exactly the domain of $f(x)$, minus the turning points. In other words,

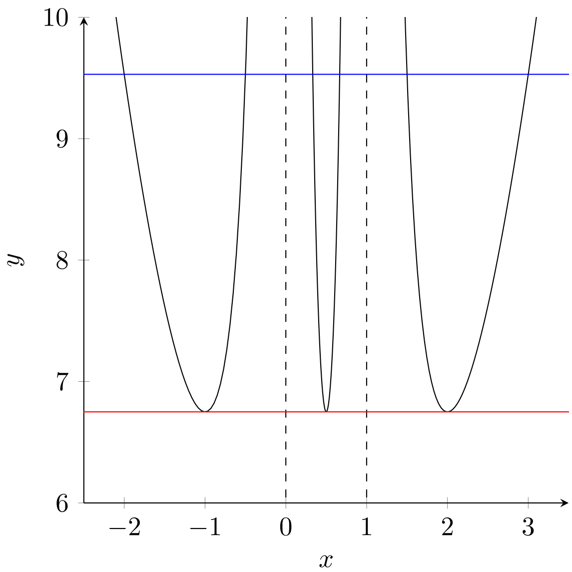
$$\{x < -1\} \cup \{-1 < x < 0\} \cup \left\{0 < x < \frac{1}{2}\right\} \cup \left\{\frac{1}{2} < x < 1\right\} \cup \{1 < x < 2\} \cup \{2 < x\}$$

Or more concisely,

$$x \in \mathbb{R} \setminus \left\{-1, 0, \frac{1}{2}, 1, 2\right\}$$

This backward slash is read '(set) minus' or '(set) difference'; $X \setminus Y$ means the set $X$ but with the elements in $Y$ removed.

Now, to find the roots of $f(x) = \frac{343}{36}$, it is probably not a good idea to just start plugging in numbers; we have a graph so we might as well use it. First, note that $\frac{343}{36} > \frac{27}{4}$. One way to see this by matching the denominators of the fractions; by multiplying the numerator and denominator of $\frac{27}{4}$ by 9, we get $\frac{243}{36}$, which is clearly less than $\frac{343}{36}$. So, we can picture $f(x) = \frac{343}{36}$ as a line higher the red line above. It is drawn on here in blue:




We can see from the graph that this equation must have six solutions. Now, note that the number $\frac{343}{36}$ has been chosen specifically for this problem, so it probably has some significance. Indeed, the numerator is a cube number ($343 = 7^3$) and the denominator is a square ($36 = 6^2$). The function $f(x)$ is also of this form, with a cube on the numerator and a square on the denominator. So we don't need to plug in random numbers in the hope of getting this obscure fraction, we can instead work out how to produce the correct value on the numerator and the denominator.

$$\frac{7^3}{6^2} = \frac{(x^2 - x + 1)^3}{(x^2 - x)^2} \implies \begin{array}{l} 7 = x^2 - x + 1 \\ 6 = x^2 - x \end{array}$$

There is no guarantee that all of the solutions will neatly produce exactly 343 on the numerator and 36 on the denominator (most likely, there will be a lot of extra terms and cancellations), but this is a good place to start to get a solution. If we can find a value of $x$ that satisfies both equations, it is a solution to $f(x) = \frac{343}{36}$. We can see that both equations are solved by $x = -2, 3$. So $f(-2) = f(3) = \frac{343}{36}$.

This is a good start, but we still need to find another four roots. We can use the properties that we proved in part (ii). If $f(x) = \frac{343}{36}$, then $f(1-x) = f\left(\frac{1}{x}\right) = \frac{343}{36}$. So $x = \frac{-1}{2}, \frac{1}{3}$ are also solutions. Note that $-2 = 1 - 3$ and $3 = 1 - (-2)$, so we can't get any extra roots by using that property here. But we can try using the $(1-x)$ property on the two new roots we found, giving $x = \frac{3}{2}, \frac{2}{3}$ as extra solutions. We have now found all six solutions. Because we know there are only six solutions, if we apply any of the known transformations ($1 - x$ or $\frac{1}{x}$) to any of the six known roots, we will get back another root that we've already found. In summary,

$$x = -2, \frac{-1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3$$

And hence the ranges of values of $x$ where $f(x) > \frac{343}{36}$ are just the ranges where the graph is above the blue line, which are

$$\{x < -2\} \cup \left\{\frac{-1}{2} < x < 0\right\} \cup \left\{0 < x < \frac{1}{3}\right\} \cup \left\{\frac{2}{3} < x < 1\right\} \cup \left\{1 < x < \frac{3}{2}\right\} \cup \{3 < x\}$$

# 6 Convergence of Sequences

If you're interested, the theorem at the start of the question is called the 'least upper bound' axiom of the real numbers, or sometimes the 'fundamental' axiom of the real numbers. This axiom is what gives the real numbers their unique character compared to the rationals. It is taught in one of the first courses in Part IA, Numbers and Sets.

(i) It might help to quickly calculate the first few terms of the sequence so we can use them later, and to provide a bit of intuition in solving the problem:

$$u_1 = 1; \quad u_2 = 1 + \frac{1}{1} = 2; \quad u_3 = 1 + \frac{1}{2} = \frac{3}{2}; \quad u_4 = 1 + \frac{2}{3} = \frac{5}{3}; \quad u_5 = 1 + \frac{3}{5} = \frac{8}{5}; \quad u_6 = 1 + \frac{5}{8} = \frac{13}{8}$$

You may recognise the numerators and denominators as increasing terms of the Fibonacci sequence, although that isn't required here in order to solve the problem. To show the result in part (i), we need to start by getting an expression for $u_{n+2}$. We can do this by plugging $(*)$ into itself.

$$u_{n+2} = 1 + \frac{1}{1 + \frac{1}{u_n}} = 1 + \frac{u_n}{u_n + 1} = \frac{2u_n + 1}{u_n + 1} \tag{$\dagger$}$$

We also need some kind of expression relating $u_{n-2}$. Since $n$ can be any number, we can substitute $n - 2$ into $(\dagger)$ to get a similar relationship for $n$ and $n - 2$.

$$u_n = \frac{2u_{n-2} + 1}{u_{n-2} + 1} \tag{$\ddagger$}$$

Now, you should start to see similarities between the above two equations and the target we are trying to prove. We can see that the left hand side of $(\dagger)$ minus the left hand side of $(\ddagger)$ gives the left hand side of our target. You might also spot the denominators of the right hand sides, which could combine to give the $(1 + u_n)(1 + u_{n-2})$ on the denominator of the right hand side of the target equation. So let us compute $(\dagger) - (\ddagger)$ and see what the right hand side becomes.

$$\begin{aligned}
u_{n-2} - u_n &= \frac{2u_n + 1}{u_n + 1} - \frac{2u_{n-2} + 1}{u_{n-2} + 1} \\
&= \frac{(2u_n + 1)(u_{n-2} + 1)}{(u_n + 1)(u_{n-2} + 1)} - \frac{(2u_{n-2} + 1)(u_n + 1)}{(u_{n-2} + 1)(u_n + 1)} \\
&= \frac{(2u_n + 1)(u_{n-2} + 1) - (2u_{n-2} + 1)(u_n + 1)}{(1 + u_n)(1 + u_{n-2})} \\
&= \frac{2u_n u_{n-2} + u_{n-2} + 2u_n + 1 - 2u_n u_{n-2} - u_n - 2u_{n-2} - 1}{(1 + u_n)(1 + u_{n-2})} \\
&= \frac{u_n - u_{n-2}}{(1 + u_n)(1 + u_{n-2})}
\end{aligned}$$

as required.

(ii) Typically when STEP says 'or otherwise', it's a bad idea to do it any other way. So let's form a base case, and then make an inductive step. We will start with $n = 1$ for simplicity, since $u_1 = 1$ so clearly the statement holds in this case. Inductively, we want to assume that $1 \le u_n \le 2$ and then to prove that $1 \le u_{n+1} \le 2$. We know that $u_{n+1} = 1 + \frac{1}{u_n}$. Since $u_n$ is at most 2, then $1 + \frac{1}{u_n}$ is at least $1 + \frac{1}{2} = \frac{3}{2}$. Also, since $u_n$ is at least 1, then $1 + \frac{1}{u_n}$ is at most $1 + \frac{1}{1} = 2$. So $\frac{3}{2} \le x \le 2$, which is a stronger condition than required. So by induction, $1 \le u_n \le 2$ for all $n$.

(iii) In order to apply the 'least upper bound' axiom, we must first show that the sequence $u_1, u_3, u_5, \ldots$ is increasing and bounded above. Clearly, the sequence is bounded above; we showed in part (ii) that every term in the sequence is bounded above by 2. To show that this sequence is increasing, we want to show that $u_{n+2} \ge u_n$ for all $n$. (In this case, $n$ is odd, but the same logic works if $n$ is even.) The sequence is characterised by the relationship we found in part (i), since we are only interested in every other term. We can see now that the left hand side of the equation, $u_{n+2} - u_n$, must always be greater than or equal to zero in order for the sequence to be increasing.

We can prove that $u_{n+2} - u_n \geq 0$, and hence that the sequence is increasing, by induction. As a base case, we can take $n = 1$, which gives $u_3 - u_1 = \frac{3}{2} - 1 \geq 0$ as required. Now, we can make an inductive step. Given that $u_n - u_{n-2} \geq 0$, we can deduce that

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1 + u_n)(1 + u_{n-2})}$$

The numerator of the right hand side is positive (by the inductive hypothesis) and the denominator is also positive because $u_n$ and $u_{n-2}$ are at least 1, by part (ii). Hence $u_{n+2} - u_n \geq 0$, and by induction this is true for all odd $n$.

We have now shown that the sequence $u_1, u_3, u_5, \ldots$ is increasing and bounded above. Therefore, by the theorem above, this sequence converges to a limit. We can apply the same logic to the sequence $u_2, u_4, u_6, \ldots$. We haven't yet seen if it is increasing, however. We will start by considering $n = 2$, and we get $u_4 - u_2 = \frac{5}{3} - 2 \leq 0$. In fact, this sequence is decreasing; we can guess this by calculating $u_6 - u_4$ and other similar expressions. We can prove this by induction, with the base case $n = 2$ as we have already shown. Then, as before,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1 + u_n)(1 + u_{n-2})}$$

The denominator is positive, as we have seen already, but this time the numerator is negative. Therefore $u_{n+2} - u_n \leq 0$, and by induction the entire sequence is decreasing.

So, we have a decreasing sequence, but how can we show it tends to a limit? Note that the theorem at the top of the question only talks about increasing sequences bounded above, but it follows that decreasing sequences bounded below also tend to a limit. Indeed, if $a_1, a_2, \ldots$ is a decreasing sequence bounded below, then $-a_1, -a_2, \ldots$ is an increasing sequence bounded above. Since $u_2, u_4, u_6, \ldots$ is bounded below by 1, using part (ii), it also converges to a limit.

Now, we can compute what these limits are. In the limit, in both the odd and even case, $u_{n+2} \to u_n$. Therefore, in the limit, we can let $u_n = u_{n+2} = u$ and substitute into $(*)$ to get

$$\begin{aligned}
u &= 1 + \frac{1}{1 + \frac{1}{u}} \\
&= 1 + \frac{1}{\frac{1+u}{u}} \\
&= 1 + \frac{u}{1 + u} \\
&= \frac{1 + u + u}{1 + u} \\
&= \frac{1 + 2u}{1 + u} \\
\therefore u(1 + u) &= 1 + 2u \\
u + u^2 &= 1 + 2u \\
u^2 - u - 1 &= 0
\end{aligned}$$

The values of $u$ which solve this equation are $u = \frac{1 \pm \sqrt{5}}{2}$. However, only one of these is in the required range, so the only solution is $u = \frac{1+\sqrt{5}}{2}$. So the sequence $u_1, u_3, u_5, \ldots$ and the sequence $u_2, u_4, u_6, \ldots$ both tend to $\frac{1+\sqrt{5}}{2}$, which for convenience we will abbreviate to $\varphi$.

Since both the odd terms and the even terms of the main sequence $u_1, u_2, u_3, \ldots$ tend to $\varphi$, the whole sequence must also tend to $\varphi$.

Now, for the last part, if $u_1 = 3$ then

$$u_2 = 1 + \frac{1}{3} = \frac{4}{3}$$

All successive terms will now lie between 1 and 2, as we have shown in part (ii), and so all of the above arguments will hold. Even if $u_1 = 3$, the sequence will still tend to $\varphi$.

# 7 Integer Equations

(i) Recall that 'non-negative' means 'zero or greater'. Zero is the easiest number to work with; we can see that $x = 1$, $y = 0$ solves this equation. Now, let us substitute $x = 3p + 4q$ and $y = 2p + 3q$ into this equation:

$$\begin{aligned}(3p+4q)^2 - 2(2p+3q)^2 &= 9p^2 + 24pq + 16q^2 - 8p^2 - 24pq - 18q^2 \\ &= p^2 - 2q^2\end{aligned}$$

But we are told that $x = p$ and $y = q$ solves $(*)$, so $p^2 - 2q^2 = 1$. Hence $(3p+4q)^2 - 2(2p+3q)^2 = 1$, and so $x = 3p+4q$ and $y = 2p+3q$ solve $(*)$. We can now use this new relationship to generate more solutions for $(*)$. Let $p = 1$ and $q = 0$, then $x = 3$ and $y = 2$ solves the equation. We can generate another solution; let $p = 3$ and $q = 2$, we find that $x = 17$ and $y = 12$ also works.

(ii) If $x$ is odd, then $x = 2m + 1$ for some integer $m$. Likewise, $y = 2n$ for some integer $n$. Then

$$\begin{aligned}(2m+1)^2 - 2(2n)^2 &= 1 \\ 4m^2 + 4m + 1 - 8n^2 &= 1 \\ 2n^2 &= m^2 + m \\ n^2 &= \frac{1}{2}m(m+1)\end{aligned}$$

as required.

(iii) Note that the right hand side can be written as a difference of two squares:

$$c^4 - a^2 = (c^2 - a)(c^2 + a)$$

Since $b$ is prime, either

1. $c^2 - a = b^3$, $c^2 + a = 1$
2. $c^2 - a = b^2$, $c^2 + a = b$
3. $c^2 - a = b$, $c^2 + a = b^2$
4. $c^2 - a = 1$, $c^2 + a = b^3$

Note that case 1 is impossible; since $c \geq 1$ and $a \geq 1$, $c^2 + a \geq 2$. Similarly, case 2 is impossible since we can make $c^2$ the subject for both equations, giving $b^2 + a = b - a$. Since $b^2 > b$, we have $a < -a$ which is a contradiction. So the two valid cases are 3 and 4. In each case, we can solve simultaneously. In case 3, we have

$$\begin{aligned}c^2 - a &= b \\ \implies a &= c^2 - b \\ \therefore c^2 + (c^2 - b) &= b^2 \\ 2c^2 &= b^2 + b \\ c^2 &= \frac{b^2+b}{2} \\ \therefore a &= \frac{b^2+b}{2} - b \\ &= \frac{b^2+b-2b}{2} \\ &= \frac{b^2-b}{2}\end{aligned}$$

In case 4, we have

$$\begin{aligned}
c^2 - a &= 1 \\
\implies a &= c^2 - 1 \\
\therefore c^2 + (c^2 - 1) &= b^3 \\
2c^2 &= b^3 + 1 \\
c^2 &= \frac{b^3+1}{2} \\
\therefore a &= \frac{b^3+1}{2} - 1 \\
&= \frac{b^3+1-2}{2} \\
&= \frac{b^3-1}{2}
\end{aligned}$$

Notice that in case 3, we have $c^2 = \frac{1}{2}b(b+1)$. This reminds us of part (ii), where $n = c$ and $m = b$. This tells us that $x = 2m + 1 = 2b + 1$ and $y = 2n = 2c$ is a solution to $(*)$.

$$(2b+1)^2 - 2(2c)^2 = 1$$

We can combine all of the parts to the question together, to see that solutions to $(*)$ can be converted into solutions to $a^2 + b^3 = c^4$. We can try our solutions for $(*)$ until we find one that works. $(*)$ is solved for $b = 1$ and $c = 1$, but this doesn't give a positive integer $a$ such that $a^2 + 1^3 = 1^4$. We can try the other solution $b = 8$ and $c = 6$, giving $a^2 + 8^3 = 6^4$. To solve for $a$, we can rearrange to get

$$a^2 = 6^4 - 8^3$$

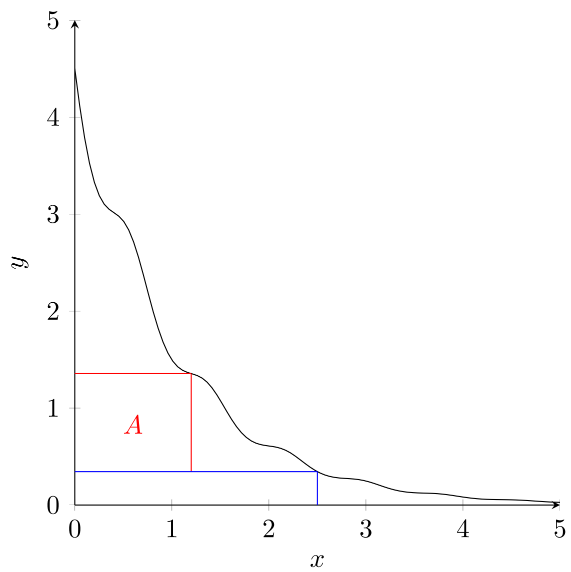
We can find that $8^3 = 512$ and $6^4 = 1296$. Then

$$\begin{aligned}
a^2 &= 1296 - 512 \\
a^2 &= 784 \\
a &= 28
\end{aligned}$$

So $(a, b, c) = (28, 6, 8)$ solves $a^2 + b^3 = c^4$.

# 8 Inequalities with Integration

(i) The area of any rectangle is the width multiplied by the height. We can see what the width and height of the rectangle is by sketching it along with any relevant lines defined by the question:



The width of the blue rectangle is $t$, and hence its height is $f(t)$. If the rectangle has width $x$, then the height of the rectangle is the height of the curve at this point, $f(x)$, minus the height of the curve at $t$, $f(t)$. So we can make an expression for $A$ as a function of $x$ and $t$:

$$A(x,t) = x(f(x) - f(t))$$

Now, we want the largest value of $A$, as $x$ varies. To find the maximum point of a function as we change a variable, we must differentiate it with respect to this variable, and set the derivative to zero to find the turning point. So we need to find the derivative of $A$ with respect to $x$ (we keep $t$ fixed), and set that to zero.

$$\frac{\mathrm{d}A}{\mathrm{d}x} = x(f'(x)) + (f(x) - f(t))$$

The value of $x$ that maximises $A$ will be called $x_0$. So we have

$$0 = x_0(f'(x_0)) + f(x_0) - f(t)$$

So $x_0$ satisfies

$$x_0(f'(x_0)) + f(x_0) = f(t)$$

The value of $A$ at $x_0$, i.e. the maximum $A$ for any given $t$, will be called $A_0(t)$. Note that this is not a function of $x$, since $A_0$ finds the value of $x$ that yields the largest area, so we are not free to choose the value of $x$ ourselves.

$$A_0(t) = A(x_0,t) = x_0(f(x_0) - f(t))$$

(ii) We can calculate this first part directly. Recall that the fundamental theorem of calculus states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int^t f(x)\,\mathrm{d}x = f(t)$$

Note that the lower limit of the integral doesn't matter, and that the variable $x$ used inside the integral is not the same as the variable $t$ used as the limit and outside the integral. We can use the product rule to find that

$$\begin{aligned}
g'(t) &= \frac{1}{t}f(t) - \frac{1}{t^2}\int_0^t f(x)\\
tg'(t) &= f(t) - \frac{1}{t}\int_0^t f(x)\\
tg'(t) &= f(t) - g(t)
\end{aligned}$$

For the next part of this question, we should consider the geometrical interpretation of the integral, since we are asked to relate it to a sketch. The integral of $f(x) - f(t)$ is the area between the lines $y = f(x)$ and $y = f(t)$. So on the sketch above, the area below the white curve and the horizontal blue line is the integral. The red rectangle $A$ is clearly inside this area for any value of $x$, so $A_0$ must be smaller than the integral.

We can now compute the integral on the left hand side of this inequality. Note that $f(t)$ is a constant since we are integrating with respect to $x$.

$$\begin{aligned}
\int_0^t (f(x) - f(t))\,\mathrm{d}x &> A_0(t)\\
\int_0^t f(x)\,\mathrm{d}t - \int_0^t f(t)\,\mathrm{d}x &> A_0(t)\\
tg(t) - tf(t) &> A_0(t)\\
-t[f(t) - g(t)] &> A_0(t)\\
-t[tg'(t)] &> A_0(t)\\
-t^2g'(t) &> A_0(t)
\end{aligned}$$

(iii) This function is a strictly decreasing function with $f(x) > 0$ for all $x \geq 0$, so we can use the properties we have proven in (i) and (ii). It looks like we will need to apply the inequality in (ii). To apply the inequality, we first need to know $A_0$. In order to find $A_0$, we need to know the value of $x_0$, so we must solve the relationship we found in part (i), which was $x_0f'(x_0) + f(x_0) = f(t)$. First, we should calculate $f'(x)$, which is

$$f'(x) = \frac{-1}{(1+x)^2}$$

Then we can write

$$\begin{aligned}
x_0\frac{-1}{(1+x_0)^2}+\frac{1}{1+x_0}&=\frac{1}{1+t}\\
\frac{-x_0}{(1+x_0)^2}+\frac{1+x_0}{(1+x_0)^2}&=\frac{1}{1+t}\\
\frac{-x_0+1+x_0}{(1+x_0)^2}&=\frac{1}{1+t}\\
\frac{1}{(1+x_0)^2}&=\frac{1}{1+t}\\
(1+x_0)^2&=1+t\\
1+x_0&=\sqrt{1+t}\\
x_0&=\sqrt{1+t}-1
\end{aligned}$$

We only need the positive square root, since $x_0$ must be positive. Now, we can find $A_0(t)$:

$$\begin{aligned}
A_0(t)&=x_0(f(x_0)-f(t))\\
&=\left(\sqrt{1+t}-1\right)\left(f\left(\sqrt{1+t}-1\right)-f(t)\right)\\
&=\left(\sqrt{1+t}-1\right)\left(\frac{1}{1+\sqrt{1+t}-1}-\frac{1}{1+t}\right)\\
&=\left(\sqrt{1+t}-1\right)\left(\frac{1}{\sqrt{1+t}}-\frac{1}{1+t}\right)\\
&=\sqrt{1+t}\cdot\frac{1}{\sqrt{1+t}}-\sqrt{1+t}\cdot\frac{1}{1+t}-\frac{1}{\sqrt{1+t}}+\frac{1}{1+t}\\
&=1-\frac{1}{\sqrt{1+t}}-\frac{1}{\sqrt{1+t}}+\frac{1}{1+t}\\
&=\frac{(1+t)-2\sqrt{1+t}+1}{1+t}\\
&=\frac{2+t-2\sqrt{1+t}}{1+t}
\end{aligned}$$

Now, we just need to find the left hand side of the equation in part (ii), $-t^2g'(t)$. So we need $g'(t)$.

$$\begin{aligned}
g'(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}\int_0^t\frac{1}{1+x}\,\mathrm{d}x\right]\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}[\ln(1+x)]_0^t\right]\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}\ln(1+t)\right]\\
&=\frac{1}{t}\cdot\frac{1}{1+t}+\frac{-1}{t^2}\cdot\ln(1+t)\\
\therefore -t^2g'(t)&=\ln(1+t)-\frac{t}{1+t}
\end{aligned}$$

Now we can substitute everything we know into the equation we found at the end of part (ii).

$$\begin{aligned}
\ln(1+t) - \frac{t}{1+t} &> \frac{2+t-2\sqrt{1+t}}{1+t} \\
\ln(1+t) &> \frac{t}{1+t} + \frac{2+t-2\sqrt{1+t}}{1+t} \\
\ln(1+t) &> \frac{2+2t-2\sqrt{1+t}}{1+t} \\
\frac{1}{2}\ln(1+t) &> \frac{1+t-\sqrt{1+t}}{1+t} \\
\ln(1+t)^{\frac{1}{2}} &> \frac{1+t}{1+t} - \frac{\sqrt{1+t}}{1+t} \\
\ln\sqrt{1+t} &> 1 - \frac{\sqrt{1+t}}{1+t}
\end{aligned}$$

# 12 Poisson Distributions

(i) If $U \sim \mathrm{Po}(\lambda)$, then we know

$$\mathrm{P}\left(U = x\right) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$$

The expectation of a non-negative discrete random variable is defined as

$$\mathrm{E}\left(X\right) = \sum_{x=0}^{\infty} x \cdot \mathrm{P}\left(X = x\right)$$

Therefore, we can calculate the expectation of $X$ as follows:

$$\begin{aligned}
\mathrm{E}\left(X\right) &= \sum_{x=0}^{\infty} x \cdot \mathrm{P}\left(X = x\right) \\
&= 1\,\mathrm{P}\left(X=1\right) + 2\,\mathrm{P}\left(X=2\right) + 3\,\mathrm{P}\left(X=3\right) + 4\,\mathrm{P}\left(X=4\right) + \cdots \\
&= 1\,\mathrm{P}\left(U=1\right) + 3\,\mathrm{P}\left(U=3\right) + 5\,\mathrm{P}\left(U=5\right) + 7\,\mathrm{P}\left(U=7\right) + \cdots \\
&= 1e^{-\lambda} \cdot \frac{\lambda^1}{1!} + 3e^{-\lambda} \cdot \frac{\lambda^3}{3!} + 5e^{-\lambda} \cdot \frac{\lambda^5}{5!} + 7e^{-\lambda} \cdot \frac{\lambda^7}{7!} + \cdots \\
&= e^{-\lambda}\left[\frac{1 \cdot \lambda^1}{1!} + \frac{3 \cdot \lambda^3}{3!} + \frac{5 \cdot \lambda^5}{5!} + \frac{7 \cdot \lambda^7}{7!} + \cdots\right] \\
&= e^{-\lambda}\left[\frac{\lambda^1}{0!} + \frac{\lambda^3}{2!} + \frac{\lambda^5}{4!} + \frac{\lambda^7}{6!} + \cdots\right] \\
&= \lambda e^{-\lambda}\left[\frac{\lambda^0}{0!} + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \frac{\lambda^6}{6!} + \cdots\right] \\
&= \alpha\lambda e^{-\lambda}
\end{aligned}$$

Similarly, we can compute the expectation of $Y$.

$$
\begin{aligned}
\mathrm{E}(Y) &= \sum_{y=0}^{\infty} y \cdot \mathrm{P}(Y=y) \\
&= 1\,\mathrm{P}(Y=1) + 2\,\mathrm{P}(Y=2) + 3\,\mathrm{P}(Y=3) + 4\,\mathrm{P}(Y=4) + \cdots \\
&= 2\,\mathrm{P}(U=2) + 4\,\mathrm{P}(U=4) + 6\,\mathrm{P}(U=6) + 8\,\mathrm{P}(U=8) + \cdots \\
&= 2e^{-\lambda} \cdot \frac{\lambda^2}{2!} + 4e^{-\lambda} \cdot \frac{\lambda^4}{4!} + 6e^{-\lambda} \cdot \frac{\lambda^6}{6!} + 8e^{-\lambda} \cdot \frac{\lambda^8}{8!} + \cdots \\
&= e^{-\lambda}\left[\frac{2 \cdot \lambda^2}{2!} + \frac{4 \cdot \lambda^4}{4!} + \frac{6 \cdot \lambda^6}{6!} + \frac{8 \cdot \lambda^8}{8!} + \cdots\right] \\
&= e^{-\lambda}\left[\frac{\lambda^2}{1!} + \frac{\lambda^4}{3!} + \frac{\lambda^6}{5!} + \frac{\lambda^8}{7!} + \cdots\right] \\
&= \lambda e^{-\lambda}\left[\frac{\lambda^1}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \frac{\lambda^7}{7!} + \cdots\right] \\
&= \beta \lambda e^{-\lambda}
\end{aligned}
$$

Note that $X+Y=U$, so we can check our answers by making sure that $\mathrm{E}(X)+\mathrm{E}(Y)=\mathrm{E}(U)$. We know that $\mathrm{E}(U)=\lambda$ since it has the Poisson distribution.

$$
\begin{aligned}
\mathrm{E}(X) + \mathrm{E}(Y) &= (\alpha+\beta)\lambda e^{-\lambda} \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda
\end{aligned}
$$

as expected.

(ii) The formula for the variance of a random variable $X$ is

$$
\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \mathrm{E}(X)^2
$$

So we need to compute the expectation of $X^2$. This involves a lot of manipulation of factorials. Whenever there is a term on the numerator that we want to cancel with a factorial on the denominator, you should consider that we might need to split up that term so that it cancels nicely. An example of this is seen on the seventh line of the derivation below. This is:

$$
\begin{aligned}
\mathrm{E}\left(X^2\right) &= \sum_{x=0}^{\infty} x^2 \cdot \mathrm{P}(X=x) \\
&= 1^2\,\mathrm{P}(X=1) + 2^2\,\mathrm{P}(X=2) + 3^2\,\mathrm{P}(X=3) + 4^2\,\mathrm{P}(X=4) + \cdots \\
&= 1^2\,\mathrm{P}(U=1) + 3^2\,\mathrm{P}(U=3) + 5^2\,\mathrm{P}(U=5) + 7^2\,\mathrm{P}(U=7) + \cdots \\
&= 1^2 e^{-\lambda} \cdot \frac{\lambda^1}{1!} + 3^2 e^{-\lambda} \cdot \frac{\lambda^3}{3!} + 5^2 e^{-\lambda} \cdot \frac{\lambda^5}{5!} + 7^2 e^{-\lambda} \cdot \frac{\lambda^7}{7!} + \cdots \\
&= e^{-\lambda}\left[\frac{1^2 \cdot \lambda^1}{1!} + \frac{3^2 \cdot \lambda^3}{3!} + \frac{5^2 \cdot \lambda^5}{5!} + \frac{7^2 \cdot \lambda^7}{7!} + \cdots\right] \\
&= \lambda e^{-\lambda}\left[\frac{1 \cdot \lambda^0}{0!} + \frac{3 \cdot \lambda^2}{2!} + \frac{5 \cdot \lambda^4}{4!} + \frac{7 \cdot \lambda^6}{6!} + \cdots\right] \\
&= \lambda e^{-\lambda}\left[\frac{(0+1) \cdot \lambda^0}{0!} + \frac{(2+1) \cdot \lambda^2}{2!} + \frac{(4+1) \cdot \lambda^4}{4!} + \frac{(6+1) \cdot \lambda^6}{6!} + \cdots\right] \\
&= \lambda e^{-\lambda}\left[\left\{\frac{0 \cdot \lambda^0}{0!} + \frac{2 \cdot \lambda^2}{2!} + \frac{4 \cdot \lambda^4}{4!} + \frac{6 \cdot \lambda^6}{6!} + \cdots\right\} + \left\{\frac{\lambda^0}{0!} + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \frac{\lambda^6}{6!} + \cdots\right\}\right] \\
&= \lambda e^{-\lambda}\left[\left\{\frac{\lambda^2}{1!} + \frac{\lambda^4}{3!} + \frac{\lambda^6}{5!} + \cdots\right\} + \left\{\frac{\lambda^0}{0!} + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \frac{\lambda^6}{6!} + \cdots\right\}\right] \\
&= \lambda e^{-\lambda}\left[\lambda\left\{\frac{\lambda^1}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots\right\} + \left\{\frac{\lambda^0}{0!} + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \frac{\lambda^6}{6!} + \cdots\right\}\right] \\
&= \lambda e^{-\lambda}\left[\lambda\beta + \alpha\right]
\end{aligned}
$$

Hence

$$\begin{aligned}\operatorname{Var}(X) &= \lambda e^{-\lambda}\left[\lambda\beta+\alpha\right]-\left(\alpha\lambda e^{-\lambda}\right)^2\\ &= \frac{\lambda\alpha+\lambda^2\beta}{e^{\lambda}}-\frac{\lambda^2\alpha^2}{(e^{\lambda})^2}\end{aligned}$$

At this point, you might guess that $e^{\lambda}=\alpha+\beta$. This turns out to be true, by considering the Taylor series of $e^{\lambda}$.

$$\begin{aligned}e^{\lambda} &= \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\\ &= \left(\frac{\lambda^0}{0!}+\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}+\frac{\lambda^6}{6!}+\cdots\right)+\left(\frac{\lambda^1}{1!}+\frac{\lambda^3}{3!}+\frac{\lambda^5}{5!}+\frac{\lambda^7}{7!}+\cdots\right)\\ &= \alpha+\beta\end{aligned}$$

This then gives

$$\operatorname{Var}(X)=\frac{\lambda\alpha+\lambda^2\beta}{\alpha+\beta}-\frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$$

We can make a similar derivation for $\operatorname{Var}(Y)$.

$$\begin{aligned}\mathrm{E}\left(Y^2\right) &= \sum_{y=0}^{\infty} y^2\cdot \mathrm{P}(Y=y)\\ &= 1^2\,\mathrm{P}(Y=1)+2^2\,\mathrm{P}(Y=2)+3^2\,\mathrm{P}(Y=3)+4^2\,\mathrm{P}(Y=4)+\cdots\\ &= 2^2\,\mathrm{P}(U=2)+4^2\,\mathrm{P}(U=4)+6^2\,\mathrm{P}(U=6)+8^2\,\mathrm{P}(U=8)+\cdots\\ &= 2^2e^{-\lambda}\cdot\frac{\lambda^2}{2!}+4^2e^{-\lambda}\cdot\frac{\lambda^4}{4!}+6^2e^{-\lambda}\cdot\frac{\lambda^6}{6!}+8^2e^{-\lambda}\cdot\frac{\lambda^8}{8!}+\cdots\\ &= e^{-\lambda}\left[\frac{2^2\cdot\lambda^2}{2!}+\frac{4^2\cdot\lambda^4}{4!}+\frac{6^2\cdot\lambda^6}{6!}+\frac{8^2\cdot\lambda^8}{8!}+\cdots\right]\\ &= \lambda e^{-\lambda}\left[\frac{2\cdot\lambda^1}{1!}+\frac{4\cdot\lambda^3}{3!}+\frac{6\cdot\lambda^5}{5!}+\frac{8\cdot\lambda^7}{7!}+\cdots\right]\\ &= \lambda e^{-\lambda}\left[\frac{(1+1)\cdot\lambda^1}{1!}+\frac{(3+1)\cdot\lambda^3}{3!}+\frac{(5+1)\cdot\lambda^5}{5!}+\frac{(7+1)\cdot\lambda^7}{7!}+\cdots\right]\\ &= \lambda e^{-\lambda}\left[\left\{\frac{1\cdot\lambda^1}{1!}+\frac{3\cdot\lambda^3}{3!}+\frac{5\cdot\lambda^5}{5!}+\frac{7\cdot\lambda^7}{7!}+\cdots\right\}+\left\{\frac{\lambda^1}{1!}+\frac{\lambda^3}{3!}+\frac{\lambda^5}{5!}+\frac{\lambda^7}{7!}+\cdots\right\}\right]\\ &= \lambda e^{-\lambda}\left[\left\{\frac{\lambda^1}{0!}+\frac{\lambda^3}{2!}+\frac{\lambda^5}{4!}+\frac{\lambda^7}{6!}+\cdots\right\}+\left\{\frac{\lambda^1}{1!}+\frac{\lambda^3}{3!}+\frac{\lambda^5}{5!}+\frac{\lambda^7}{7!}+\cdots\right\}\right]\\ &= \lambda e^{-\lambda}\left[\lambda\left\{\frac{\lambda^0}{0!}+\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}+\frac{\lambda^6}{6!}+\cdots\right\}+\left\{\frac{\lambda^1}{1!}+\frac{\lambda^3}{3!}+\frac{\lambda^5}{5!}+\frac{\lambda^7}{7!}+\cdots\right\}\right]\\ &= \lambda e^{-\lambda}\left[\lambda\alpha+\beta\right]\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{Var}(Y) &= \lambda e^{-\lambda}\left[\lambda\alpha+\beta\right]-\left(\beta\lambda e^{-\lambda}\right)^2\\ &= \frac{\lambda^2\alpha+\lambda\beta}{\alpha+\beta}-\frac{\lambda^2\beta^2}{(\alpha+\beta)^2}\end{aligned}$$

For the final part of the question, we need $\operatorname{Var}(X+Y)$. But since $X+Y=U$, $\operatorname{Var}(X+Y)=\operatorname{Var}(U)=\lambda$.

So we need to solve

$$\begin{aligned}
\lambda &= \mathrm{Var}\,(X) + \mathrm{Var}\,(Y) \\
\lambda &= \frac{\lambda\alpha + \lambda^2\beta}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2} + \frac{\lambda^2\alpha + \lambda\beta}{\alpha+\beta} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2} \\
1 &= \frac{\alpha + \lambda\beta}{\alpha+\beta} - \frac{\lambda\alpha^2}{(\alpha+\beta)^2} + \frac{\lambda\alpha + \beta}{\alpha+\beta} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2} \\
1 &= \frac{\alpha + \lambda\beta + \lambda\alpha + \beta}{\alpha+\beta} - \frac{\lambda\alpha^2 + \lambda\beta^2}{(\alpha+\beta)^2} \\
1 &= (1+\lambda)\frac{\alpha+\beta}{\alpha+\beta} - \lambda\frac{\alpha^2+\beta^2}{(\alpha+\beta)^2} \\
1 &= 1 + \lambda - \lambda\frac{\alpha^2+\beta^2}{(\alpha+\beta)^2} \\
\lambda &= \lambda\frac{\alpha^2+\beta^2}{(\alpha+\beta)^2}
\end{aligned}$$

Since $\lambda$ is non-zero,

$$\begin{aligned}
1 &= \frac{\alpha^2+\beta^2}{(\alpha+\beta)^2} \\
(\alpha+\beta)^2 &= \alpha^2+\beta^2 \\
\alpha^2 + 2\alpha\beta + \beta^2 &= \alpha^2+\beta^2 \\
2\alpha\beta &= 0
\end{aligned}$$

which cannot happen for $\lambda \neq 0$. So there are no non-zero values of $\lambda$ for which $\mathrm{Var}\,(X) + \mathrm{Var}\,(Y) = \mathrm{Var}\,(X+Y)$.

# 13 Biased Coin

(i) If $A = 1$, then the longest alternating run had length 1, so the second coin flip must be the same result as the first coin flip. So we either got two heads (with probability $p^2$) or two tails (with probability $q^2$). So the total probability is just the sum, $p^2 + q^2$. If $S = 1$, then the longest straight run had length 1, so the second flip must be a different result to the first flip, meaning we either got a head then a tail, or a tail then a head. These events have probability $pq$ and $qp$, giving a total of $\mathrm{P}\,(S = 1) = 2pq$.

You might recognise $p^2 + q^2$ and $2pq$ as terms in the expansion of $(p+q)^2$ or $(p-q)^2$; this is probably the easiest way to do this part of the question if you see this relationship. In this question, it turns out that we need to use $(p-q)^2$. Note that $(p-q)^2$ is always positive, since it is a square number. Then

$$\begin{aligned}
(p-q)^2 &> 0 \\
p^2 - 2pq + q^2 &> 0 \\
p^2 + q^2 &> 2pq \\
\mathrm{P}\,(A = 1) &> \mathrm{P}\,(S = 1)
\end{aligned}$$

as required. If we had used the inequality $(p+q)^2 > 0$ instead, we wouldn't have been able to make this deduction. The $(p-q)^2$ expression worked because there were some negative terms that we could move to the other side of the inequality.

(ii) For convenience, we will abbreviate the results of coin tosses with the initials H (heads) and T (tails), so a run of two heads then a tail will be written HHT. If $S = 2$, we must have tossed HHT or TTH. The total probability is

$$\mathrm{P}\,(S = 2) = \mathrm{P}\,(\mathrm{HHT}) + \mathrm{P}\,(\mathrm{TTH}) = p^2q + q^2p = pq(p+q)$$

If $A = 2$, then we either tossed HTT or THH.

$$\mathrm{P}\,(A = 2) = \mathrm{P}\,(\mathrm{HTT}) + \mathrm{P}\,(\mathrm{THH}) = pq^2 + qp^2 = pq(p+q) = \mathrm{P}\,(S = 2)$$

as required. For $S = 3$, we have

$$\mathrm{P}(S=3) = \mathrm{P}(\mathrm{HHHT}) + \mathrm{P}(\mathrm{TTTH}) = p^3q + q^3p = pq(p^2+q^2)$$

For $A = 3$, we have

$$\mathrm{P}(A=3) = \mathrm{P}(\mathrm{HTHH}) + \mathrm{P}(\mathrm{THTT}) = p^3q + q^3p = pq(p^2+q^2) = \mathrm{P}(S=3)$$

(iii) We need to compute $\mathrm{P}(S=2n)$ and $\mathrm{P}(A=2n)$. If $S = 2n$, then we rolled heads $2n$ times and then tails, or we rolled tails $2n$ times and then heads.

$$\mathrm{P}(S=2n) = \mathrm{P}\left(\mathrm{H}^{2n}\mathrm{T}\right) + \mathrm{P}\left(\mathrm{T}^{2n}\mathrm{H}\right) = p^{2n}q + q^{2n}p = pq(p^{2n-1}+q^{2n-1})$$

If $A = 2n$, then we rolled heads and tails $n$ times each, followed by either heads or tails (depending on which way the first coin landed).

$$\mathrm{P}(A=2n) = \mathrm{P}(\mathrm{H}^n\mathrm{T}^n\mathrm{H}) + \mathrm{P}(\mathrm{H}^n\mathrm{T}^n\mathrm{T}) = p^nq^np + p^nq^nq = p^nq^n(p+q)$$

To work out which is larger, we can subtract one from the other and consider the sign of the result.

$$\begin{aligned}
\mathrm{P}(S=2n) - \mathrm{P}(A=2n) &= pq(p^{2n-1}+q^{2n-1}) - p^nq^n(p+q) \\
&= pq\left(p^{2n-1}+q^{2n-1} - p^{n-1}q^{n-1}(p+q)\right) \\
&= pq\left(p^{2n-1} - p^nq^{n-1} + q^{2n-1} - p^{n-1}q^n\right) \\
&= pq\underbrace{\left(p^{n-1}-q^{n-1}\right)}_{\alpha}\underbrace{\left(p^n-q^n\right)}_{\beta}
\end{aligned}$$

If $p > q$, then $\alpha > 0$ and $\beta > 0$. If $p < q$, then $\alpha < 0$ and $\beta < 0$. Either way, $\mathrm{P}(S=2n) - \mathrm{P}(A=2n)$ is positive, as required. Now, if $S = 2n+1$:

$$\mathrm{P}(S=2n+1) = \mathrm{P}\left(\mathrm{H}^{2n+1}\mathrm{T}\right) + \mathrm{P}\left(\mathrm{T}^{2n+1}\mathrm{H}\right) = p^{2n+1}q + q^{2n+1}p = pq(p^{2n}+q^{2n})$$

If $A = 2n+1$, then we rolled $n+1$ heads, $n$ tails, and a final head; or $n+1$ tails, $n$ heads, and a final tail.

$$\mathrm{P}(A=2n+1) = \mathrm{P}\left(\mathrm{H}^n\mathrm{T}^n\mathrm{H}^2\right) + \mathrm{P}\left(\mathrm{H}^n\mathrm{T}^n\mathrm{T}^2\right) = p^nq^np^2 + p^nq^nq^2 = p^nq^n(p^2+q^2)$$

Again, we will compute $\mathrm{P}(S=2n+1) - \mathrm{P}(A=2n+1)$.

$$\begin{aligned}
\mathrm{P}(S=2n+1) - \mathrm{P}(A=2n+1) &= pq(p^{2n}+q^{2n}) - p^nq^n(p^2+q^2) \\
&= pq\left(p^{2n}+q^{2n} - p^{n-1}q^{n-1}(p^2+q^2)\right) \\
&= pq\left(p^{2n} - p^{n+1}q^{n-1} + q^{2n} - p^{n-1}q^{n+1}\right) \\
&= pq\underbrace{\left(p^{n-1}-q^{n-1}\right)}_{\alpha}\underbrace{\left(p^{n+1}-q^{n+1}\right)}_{\beta}
\end{aligned}$$

If $p > q$, then $\alpha > 0$ and $\beta > 0$. If $p < q$, then $\alpha < 0$ and $\beta < 0$. Either way, $\mathrm{P}(S=2n+1) - \mathrm{P}(A=2n+1)$ is positive. Hence $\mathrm{P}(S=2n+1) > \mathrm{P}(A=2n+1)$.